Lecture 6-1: OLS with Matrix Algebra

Goal: Derive OLS estimator in matrix form and study its properties.

- Preparation
  - Matrix representation of regression
  - Solving systems of equations
- Linear model in matrix form
  - Vector derivatives
  - Derivation of OLS estimator
  - OLS assumptions in matrix notation
  - Properties of OLS estimator
- Frisch-Waugh-Lovell Theorem
  - Projection and annihilator matrices
  - Proving FWL

Worksheet

1. With $n = 3$, regress y on an intercept and x where

$$\mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} 1 & 1 \\ 1 & 3 \\ 1 & 2 \end{bmatrix}.$$

   (a) Compute $\hat{\boldsymbol{\beta}}$, fitted values $\hat{\mathbf{y}}$, residuals $\hat{\mathbf{u}}$.

   (b) Verify the normal equations $\mathbf{X}'\hat{\mathbf{u}} = \mathbf{0}$.

   (c) Compute $SSR = \hat{\mathbf{u}}'\hat{\mathbf{u}}$ and $\hat{\sigma}^2 = SSR/(n - (k + 1))$.

2. Under homoskedasticity and independent sampling, $V[\mathbf{u}|\mathbf{X}] = \sigma^2\mathbf{I}_n$.

   (a) Prove that $V[\hat{\boldsymbol{\beta}}|\mathbf{X}] = \sigma^2(\mathbf{X}'\mathbf{X})^{-1}$.

   (b) For any fixed $(k+1)\times 1$ vector $\mathbf{a}$, *derive* $V[\mathbf{a}'\hat{\boldsymbol{\beta}}|\mathbf{X}]$ and the plug-in standard error.

   (c) Return to the data in Question 1 and report the estimated variance–covariance matrix $\widehat{V}[\hat{\boldsymbol{\beta}}] = \hat{\sigma}^2(\mathbf{X}'\mathbf{X})^{-1}$.

3. Suppose the true model is $y_i = \mathbf{x}_i\boldsymbol{\beta} + u_i$ with $E[u_i|\mathbf{x}_i] = 0$, and the sample $\{(y_i, \mathbf{x}_i)\}_{i=1}^n$ is i.i.d. with $E[\mathbf{x}_i'\mathbf{x}_i]$ finite and nonsingular.

   (a) Show that $\hat{\boldsymbol{\beta}}$ is a consistent estimator of $\boldsymbol{\beta}$.

   (b) Assume further that $E[u_i^2|\mathbf{x}_i] = \sigma^2 < \infty$, derive the asymptotic distribution of $\hat{\boldsymbol{\beta}}$.

4. Assume the classical linear regression model satisfying Assumptions I–V:

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{u},$$

where $\mathbf{X}$ is an $n \times k$ matrix of full column rank, and $\mathbf{y}$ is an $n \times 1$ vector of responses.

(a) What does $(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$ represent in the context of OLS estimation?

(b) What is the interpretation of $\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$?

(c) What does $\mathbf{y} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$ represent?

(d) What does $[\mathbf{y} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}]'[\mathbf{y} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}]$ compute?

(e) What does $[\mathbf{y} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}]'[\mathbf{y} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}]/(n-k)$ estimate?

(f) What is the interpretation of

$$(\mathbf{X}'\mathbf{X})^{-1}\,\frac{[\mathbf{y} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}]'[\mathbf{y} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}]}{(n-k)}\,?$$

5. Following the same setup in the previous question, define:

$$\mathbf{Q} = \mathbf{X}'\mathbf{X}, \qquad \mathbf{A} = \mathbf{Q}^{-1}\mathbf{X}', \qquad \mathbf{P} = \mathbf{X}\mathbf{A}, \qquad \mathbf{M} = \mathbf{I} - \mathbf{P}.$$

Here, $\mathbf{P}$ is the projection (hat) matrix and $\mathbf{M}$ is the residual maker matrix.

Hint: Many of the results in this problem build on earlier parts. Use previously shown equalities (e.g., $\mathbf{P}^2 = \mathbf{P}$, $\mathbf{P}' = \mathbf{P}$, $\mathbf{M} = \mathbf{I} - \mathbf{P}$) to simplify the algebra.

(a) Express the residuals from regressing $\mathbf{y}$ on $\mathbf{X}$ as a function of $\mathbf{M}$.

(b) Show that $\mathbf{P}$ is symmetric and idempotent.

(c) Show that $\mathbf{M}$ is idempotent (verify $\mathbf{M}'\mathbf{M} = \mathbf{M}$ and $\mathbf{M}^2 = \mathbf{M}$).

(d) Show that $\mathbf{AP} = \mathbf{A}$, $\mathbf{AM} = \mathbf{0}$, $\mathbf{MP} = \mathbf{0}$, and $\mathbf{PM} = \mathbf{0}$.

(e) Show that $\mathbf{PX} = \mathbf{X}$, $\mathbf{MX} = \mathbf{0}$.

(f) Show that $\mathbf{P}\,\widehat{\mathbf{y}} = \widehat{\mathbf{y}}$, $\mathbf{P}\,\widehat{\mathbf{u}} = \mathbf{0}$.

(g) Show that $\mathbf{M}\,\widehat{\mathbf{y}} = \mathbf{0}$, $\mathbf{M}\,\widehat{\mathbf{u}} = \widehat{\mathbf{u}}$.

(h) Show that $\mathbf{X}'\widehat{\mathbf{y}} = \mathbf{X}'\mathbf{y}$.

(i) Show that $\widehat{\mathbf{u}}'\widehat{\mathbf{u}} = \mathbf{y}'\mathbf{M}\mathbf{y} = \mathbf{y}'\mathbf{y} - \widehat{\mathbf{y}}'\widehat{\mathbf{y}}$.